A note on completeness of weighted Banach spaces of analytic functions

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Abstract

Given a non-negative weight $v$, not necessarily bounded or strictly positive, defined on a domain $G$ in the complex plane, we consider the weighted space $H^\infty_v(G)$ of all holomorphic functions on $G$ such that the product $v|f|$ is bounded in $G$ and study the question of when is such a space complete. We obtain both some necessary and some sufficient conditions, exhibit several relevant examples, and characterize completeness in the case of spaces with radial weights on balanced domains.

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1 Introduction, Notation, and Motivation

In this paper, as is usual, by a domain we will mean an open connected set in the complex plane \( \mathbb{C} \). A weight \( v \) on a domain \( G \) is a non-negative function \( v : G \to [0, \infty[ \). In general, we shall not require that \( v \) be bounded or strictly positive. Denote by \( H(G) \) the space of all holomorphic functions on \( G \) and by \( \tau_{co} \) the topology of uniform convergence on the compact subsets of \( G \) (often also called the compact-open topology). It is not difficult to check that the space \( (H(G), \tau_{co}) \) is a metrizable and complete locally convex space, i.e. a Fréchet space.

Given a weight \( v \) on \( G \), we will use the following notation:

\[
E_v := \{ z \in G \mid v(z) > 0 \}.
\]

The weighted space of holomorphic functions \( H^\infty_v(G) \) associated with \( v \) is defined by

\[
H^\infty_v(G) := \{ f \in H(G) \mid \| f \|_v = \sup_{z \in \mathbb{D}} v(z) |f(z)| < +\infty \}
\]

and is endowed with the natural seminorm \( \| f \|_v := \sup_{z \in \mathbb{D}} v(z) |f(z)| \). Spaces of this type when \( v \) is strictly positive and continuous appear in the study of growth conditions of analytic functions and have been investigated in various articles since the work of Shields and Williams; see, e.g., [3], [4], [6], [10], [11], [13] and the references therein.

If \( v \) is the constant function 1, then \( H^\infty_v(G) \) obviously coincides with the space \( H^\infty(G) \) of all bounded holomorphic functions on \( G \) endowed with the sup-norm \( \| \cdot \|_\infty \). In fact, in most cases considered in the literature, \( v \) is strictly positive. In this case it is rather clear that the above weighted space is complete. It might be somewhat less obvious that if this is not required then the space may actually not be complete. This is easily seen by considering a weight that vanishes on a “sizeable” portion of the disk; for example, \( v(z) = \max\{0, \Re z\} \) is continuous, vanishes in the left-hand half of the disk, and is strictly positive in the remaining open right semi-disk. It is easy to construct a sequence of functions in the space which blow up near \( z = -1 \) and are bounded elsewhere and to see that they form a Cauchy sequence; however, such a sequence cannot possibly have a subsequence that converges uniformly on compact subsets of the disk.

The problem we consider in this note is the following: When is the space \( H^\infty_v(G) \) complete? In other words, when is it a Banach space? Proposition 2.3, whose content should be intuitively clear to the experts, translates this into other terms. A general characterization which might be of further interest is provided by Theorem 2.5. Several necessary as well as sufficient conditions are given by Propositions 2.6, 2.8, 2.9 and 2.10. They are used to present concrete examples in Corollaries 2.11, 2.12 and 2.13 and Proposition 2.14. However, we are presently not able to give a complete answer to this question.

The situation is similar in other related function spaces. For example, the completeness of weighted Bergman spaces was studied by Arcozzi and Björn [1]. They obtained complete characterizations when the weight \( v(z) = \chi_E(z), z \in G \), coincides with the characteristic function \( \chi_E \) of a subset \( E \) of \( G \) in [1, Theorem 2.1]. Partial results concerning weighted Bergman spaces \( A^p_v(G), 1 \leq p < \infty \), for a positive Borel measure \( \mu \) on \( G \) are given in
Section 5 of [1]. That research was taken up by Björn in a different direction [5]. Nakazi [12] investigated the completeness of weighted Bloch spaces, that is a question related to the problem we investigate here.

2 Results

For most reasonable weights our weighted space is complete and one certainly expects it to be normed. However, even this is not always the case.

Proposition 2.1 Let \( v : G \to [0, \infty[ \) be a weight on a domain \( G \subset \mathbb{C} \). Then the space \( H_v^\infty(G) \) is normed if and only if \( E_v = \{ z \in G \mid v(z) > 0 \} \) has a limit point in \( G \).

Proof. If \( E_v \) has a limit point in \( G \), then the seminorm \( \| \cdot \|_v \) is a norm by the uniqueness principle for holomorphic functions. Conversely, if \( E_v \) does not have a limit point in \( G \), we can apply Weierstrass interpolation theorem (see e.g. [2, Theorem 3.3.1]) to get a non-zero holomorphic function \( f \in H(G) \) such that \( f(z) = 0 \) for each \( z \in E_v \). Then \( \| f \|_v = 0 \) and \( f \neq 0 \), hence \( \| \cdot \|_v \) is not a norm.

Now that this elementary issue has been settled, we turn to the completeness question. We first need a lemma.

Lemma 2.2 Let \( v : G \to [0, \infty[ \) be a weight on a domain \( G \subset \mathbb{C} \). If the space \( H_v^\infty(G) \) is normed, then the inclusion map \( J : H_v^\infty(G) \to (H(G), \tau_{co}) \) has closed graph.

Proof. Let \( (f_j)_j \) be a sequence in \( H_v^\infty(G) \) such that \( f_j \to f \) in \( H_v^\infty(G) \) and \( f_j \to g \) in \( (H(G), \tau_{co}) \) as \( j \to \infty \). In particular, \( f_j(z) \to f(z) \) as \( j \to \infty \) for each \( z \in E_v \), and \( f_j(z) \to g(z) \) as \( j \to \infty \) for each \( z \in G \). Then \( f \) and \( g \) are two holomorphic functions on \( G \) which coincide on the set \( E_v \), that has a limit point in \( G \) by Proposition 2.1. By the uniqueness principle for holomorphic functions, \( f = g \) on \( G \) and we are done.

For a point \( z \in G \), we denote by \( \delta_z : H(G) \to \mathbb{C} \) the (linear) point evaluation functional \( \delta_z(f) := f(z) \), \( f \in H(G) \), as well as its restriction to \( H_v^\infty(G) \). When \( H_v^\infty(G) \) is normed, the norm of \( H_v^\infty(G)' \) will be denoted by \( \| \cdot \|_v \). The following result summarizes a result an expert would expect: the completeness of our space is essentially equivalent to the boundedness of the point evaluation functionals. Regarding condition (iv) below (uniform boundedness of point evaluations on compact sets), it should be pointed out that this property is in turn equivalent to their boundedness at each point in view of the uniform boundedness principle.

Proposition 2.3 Assume that the space \( H_v^\infty(G) \) is normed. The following conditions are equivalent:

(i) The space \( H_v^\infty(G) \) is a Banach space.

(ii) The inclusion map \( J : H_v^\infty(G) \to (H(G), \tau_{co}) \) is continuous.
(iii) The closed unit ball $B_v^\infty$ of $H_v^\infty(G)$ is bounded in $(H(G), \tau_{co})$.

(iv) For each $z \in G \delta_z \in H_v^\infty(G)'$, and $\sup_{z \in K} \|\delta_z\|'_v < \infty$ for each compact subset $K$ of $G$.

**Proof.** Condition (i) implies condition (ii) as a consequence of Lemma 2.2 and the closed graph theorem for Fréchet spaces.

To prove that condition (ii) implies condition (i), fix a Cauchy sequence $(f_j)_j$ in $H_v^\infty(G)$. By the assumption (ii), there is $f \in H(G)$ such that $(f_j)_j$ converges to $f$ uniformly on the compact subsets of $G$. On the other hand,

$$\forall \varepsilon > 0 \exists J \forall j \geq J \forall z \in G : v(z) |f_j(z) - f_k(z)| < \varepsilon.$$  

If $v(z) = 0$, then $v(z) |f_j(z) - f(z)| = 0$, $j \geq J$, and if $v(z) > 0$, letting $k \to \infty$, $v(z) |f_j(z) - f(z)| \leq \varepsilon$ for all $j \geq J$. This implies, for $\varepsilon = 1$, $v(z) |f(z)| \leq 1 + \|f_j\|_v$ for each $z \in G$ and $f \in H_v^\infty(G)$. Moreover, for arbitrary $\varepsilon$, $f_j \to f$ in $H_v^\infty(G)$ as $j \to \infty$.

Thus, conditions (i) and (ii) are equivalent. Clearly, conditions (ii) and (iii) are also equivalent.

Suppose now that condition (ii) holds. Since $\delta_z \in (H(G), \tau_{co})'$ for each $z \in G$, it follows that $\delta_z \in H_v^\infty(G)'$ for each $z \in G$. Moreover, given a compact set there is $C > 0$ such that

$$\sup_{z \in K} |f(z)| \leq C \sup_{z \in G} v(z) |f(z)|$$

for each $f \in H_v^\infty(G)$. This implies $\|\delta_z\|'_v \leq C$ for each $z \in K$. Hence condition (iv) follows.

Finally, assume that condition (iv) holds. Fix a compact set $K$ in $G$ and set $M := \sup_{z \in K} \|\delta_z\|'_v$. If $f \in H_v^\infty(G)$ satisfies $\|f\|_v \leq 1$, then $|f(z)| \leq \|\delta_z\|'_v \leq M$ for each $z \in K$. This implies $\sup_{z \in K} |f(z)| \leq M \|f\|_v$ for each $f \in H_v^\infty(G)$, and the inclusion map $J : H_v^\infty(G) \to (H(G), \tau_{co})$ is continuous. \hfill \Box

**Remark 2.4** As a consequence of Ptak’s version of the closed graph theorem [9, Theorem 4, page 301], if $H_v^\infty(G)$ is a non-complete normed space, then it is not barrelled, i.e., there are weak-* bounded sets in the topological dual which are not norm bounded.

**Theorem 2.5** Let $v : G \to [0, \infty]$ be a bounded weight on a domain $G \subset \mathbb{C}$ such that the space $H_v^\infty(G)$ is normed. The space $H_v^\infty(G)$ is complete if and only if there is a bounded, continuous, strictly positive weight $\tilde{v}$ on $G$ such that $H_v^\infty(G) = H_{\tilde{v}}^\infty(G)$.

**Proof.** It is well-know that if $\tilde{v}$ is a bounded, continuous, strictly positive weight $\tilde{v}$ on $G$, then the space $H_{\tilde{v}}^\infty(G)$ is a Banach space. We prove the converse. To do this we follow ideas of [4]. By assumption there is $M > 0$ such that $0 \leq v(z) \leq M$ for each $z \in G$, hence the constant function $f_0(z) := 1/M$, $z \in G$, belongs to $H_v^\infty(G)$ and $\|f_0\|_v \leq 1$.

For each $z \in G$ we have $\delta_z \in H_v^\infty(G)'$ by Proposition 2.3, and $\|\delta_z\|'_v \geq |f_0(z)| = 1/M > 0$. We set

$$\tilde{v}(z) := 1/\|\delta_z\|'_v, \ z \in G.$$
By our estimate above $0 < \tilde{v}(z) \leq M$ for each $z \in G$. Moreover, $v(z) \leq \tilde{v}(z)$ for each $z \in G$. In fact, the inequality is obvious if $v(z) = 0$. If $v(z) > 0$ and $g \in H_v^\infty(G)$ satisfies $\|g\|_v \leq 1$, then $|g(z)| \leq 1/v(z)$. This implies $1/\tilde{v}(z) \leq 1/v(z)$. Thus $v(z) \leq \tilde{v}(z)$. This implies in particular $H_v^\infty(G) \subset H_v^\infty(G)$ with continuous, norm decreasing inclusion.

Now $H_v^\infty(G) = H_v^\infty(G)$ holds isometrically. Indeed, given $f \in H_v^\infty(G)$ with $\|f\|_v \leq 1$, then $\tilde{v}(z)|f(z)| \leq 1$ for each $z \in G$. Therefore $f \in H_v^\infty(G)$ and $\|f\|_{\tilde{v}} \leq 1$. This implies $H_v^\infty(G) \subset H_v^\infty(G)$ with norm decreasing inclusion.

It remains to prove that the weight $\tilde{v}$ is continuous. Indeed, the map $\Delta : G \to H_v^\infty(G)^\prime$, $\Delta(z) := \delta_z$ is well defined and locally bounded since every $z \in G$ has a compact neighborhood and the conclusion follows from condition (iv) in Proposition 2.3. Now, for each $f \in H_v^\infty(G) \subset H_v^\infty(G)^\prime$, the map $T_f \circ \Delta : G \to \mathbb{C}, z \to f(z)$, is holomorphic on $G$. By [7, Theorem 1] the vector valued map $\Delta : G \to H_v^\infty(G)^\prime$ is holomorphic, hence continuous for the dual norm $\|\|_{\prime}$ on $H_v^\infty(G)^\prime$. Since the norm is continuous, it follows that the function given by $\tilde{v}(z) = 1/\|\Delta(z)\|_{\prime}$ is continuous. This completes the proof. \hfill \square

A weight $v$ on $G$ is bounded if and only if the constant functions 1 belongs to $H_v^\infty(G)$ if and only if every bounded analytic function on $G$ belongs to $H_v^\infty(G)$.

Let $A$ be subset of a domain $G$ in $\mathbb{C}$. The holomorphically convex hull of $A$ in $G$ is the set

$$Hco(A) := \{ z \in G \mid \|f(z)\| \leq \sup_{\zeta \in A} |f(\zeta)| \ \forall f \in H(G) \}.$$ 

Every open domain $G$ in $\mathbb{C}$ is holomorphically convex in the sense that for each compact set $K$ in $G$ the holomorphic convex hull $Hco(K)$ is compact and contained in $G$. See [8].

**Proposition 2.6** Let $v : G \to [0, \infty[$ be a bounded weight on a domain $G \subset \mathbb{C}$. If $H_v^\infty(G)$ is a Banach space, then $G$ coincides with the holomorphic convex hull $Hco(E_v)$ of $E_v := \{ z \in G \mid v(z) > 0 \}$.

**Proof.** We give a proof by reduction to absurd. Assume there exist a point $z_0 \in G$ and a function $g_0 \in H(G)$ such that

$$|g_0(z_0)| > \alpha > \sup_{\zeta \in E_v} |g_0(\zeta)|$$

for some $\alpha > 0$. Set $g := g_0/\alpha \in H(G)$. Then $|g(z_0)| > 1$ and $|g(\zeta)| \leq 1$ for each $\zeta \in E_v$.

We show that the sequence $(g^k)_k$ is bounded in $H_v^\infty(G)$. Since $v$ is bounded, there is $M > 0$ with $v(z) \leq M$ for each $z \in G$. If $z \notin E_v$, then $v(z)|g^k(z)| = 0$ for each $k \in \mathbb{N}$. On the other hand, if $z \in E_v$, then $v(z)|g(z)|^k \leq |g(z)|^k \leq M$. Hence $\sup_{k \in \mathbb{N}} \sup_{z \in G} v(z)|g(z)|^k \leq M$.

By assumption $H_v^\infty(G)$ is a Banach space, hence $(g^k)_k$ is a bounded sequence in $(H(G), \tau_{co})$ by Proposition 2.3 (ii). However, $|g(z_0)|^k \to \infty$ as $k \to \infty$. A contradiction. \hfill \square
Corollary 2.7 Let \( v : G \to [0, \infty[ \) be a bounded weight on a domain \( G \subset \mathbb{C} \).

(1) If \( E_v \) is contained in a convex set \( A \subset G \) which is closed in \( G \) and is not \( G \) itself, then \( H_v^\infty(G) \) is not a Banach space.

(2) If the closure of \( E_v \) in \( \mathbb{C} \) is compact and contained in \( G \), then \( H_v^\infty(G) \) is not a Banach space.

Proof. In both cases the holomorphic convex hull \( Hco(E_v) \) of \( E_v \) is strictly contained in \( G \). The conclusion follows from Proposition 2.6. \( \square \)

As is usual, we will denote the boundary of a set \( A \) by \( \partial A \).

Proposition 2.8 Let \( v : G \to [0, \infty[ \) be a bounded weight on a domain \( G \neq \mathbb{C} \). If \( H_v^\infty(G) \) is a Banach space, then the boundary \( \partial G \) is contained in the closure \( \overline{E_v} \) of \( E_v \) in \( \mathbb{C} \).

Proof. Assume there is \( z_0 \in \partial G \setminus \overline{E_v} \). We can find \( d > 0 \) such that \( v(z) = 0 \) for each \( z \in G \) with \( |z - z_0| \leq d \). Clearly \( f_k(z) := (d/(z - z_0))^k \) defines a holomorphic function in \( G \) for each \( k \in \mathbb{N} \). There is \( M > 0 \) with \( v(z) \leq M \) for each \( z \in G \). If \( z \in G \) and \( |z - z_0| \leq d \), then \( v(z)f_k(z) = 0 \). On the other hand, if \( z \in G \) and \( |z - z_0| > d \), then \( v(z)f_k(z) \leq v(z) \leq M \). This implies that the sequence \( (f_k)_k \) is bounded in \( H_v^\infty(G) \), hence in \( (H(G), \tau_{co}) \) by Proposition 2.3 (ii). Since \( z_0 \in \partial G \), there is \( x \in G \) such that \( |x - z_0| < d/2 \), and we have \( |f_k(x)| = |(d/(x - z_0))^k| > 2^k \) for each \( k \in \mathbb{N} \). A contradiction. \( \square \)

Proposition 2.9 Let \( v : G \to [0, \infty[ \) be a weight on a domain \( G \subset \mathbb{C} \). Assume that for each compact set \( K \subset G \) there is a compact set \( L \) such that \( K \subset L \subset G \), and there is a positive constant \( \alpha \) such that

\[
K \subset Hco\{ z \in L \mid v(z) \geq \alpha \}.
\]

Then \( H_v^\infty(G) \) is a Banach space.

Proof. By Proposition 2.3 (iii) it is enough to show that the closed unit ball \( B_v^{\infty} \) of \( H_v^\infty(G) \) is bounded in \( (H(G), \tau_{co}) \). To see this, fix a compact set \( K \subset G \) and apply the assumption to find the compact set \( L \subset G \) and \( \alpha > 0 \). Set \( R := \{ z \in L \mid v(z) \geq \alpha \} \). If \( f \in B_v^{\infty} \) and \( z \in K \), then \( z \in Hco(\overline{R}) \). Thus, since \( f \) is continuous and \( \overline{R} \subset L \subset G \), we get

\[
|f(z)| \leq \sup_{\zeta \in R} |f(\zeta)| = \sup_{\zeta \in R} |f(\zeta)|, \quad \forall z \in K.
\]

If \( \zeta \in R \), then \( v(\zeta) \geq \alpha \). This implies \( \alpha|f(\zeta)| \leq v(\zeta)|f(\zeta)| \leq 1 \) for each \( \zeta \in R \). Hence \( \sup_{\zeta \in R} |f(\zeta)| \leq 1/\alpha \). Therefore \( |f(z)| \leq 1/\alpha \) for each \( z \in K \) and each \( f \in B_v^{\infty} \), which implies that \( B_v^{\infty} \) is bounded in \( (H(G), \tau_{co}) \). \( \square \)
Proposition 2.10 Let \( v : G \to [0, \infty[ \) be a weight on a domain \( G \subset \mathbb{C} \). Assume that for each compact set \( K \subset G \) there is a compact set \( L \) such that \( K \subset L \subset G \), and there is \( \alpha > 0 \) such that

\[
\partial L \subset \{ z \in G \mid v(z) \geq \alpha \}.
\]

Then \( H_v^\infty(G) \) is a Banach space.

**Proof.** Fix a compact set \( K \subset G \). We apply the assumption to find a compact set \( L \) containing \( K \) and \( \alpha \) such that \( \partial L \subset \{ z \in G \mid v(z) \geq \alpha \} \). If \( G = \mathbb{C} \), take \( d = 1 \), and if \( G \neq \mathbb{C} \), set \( d := \text{dist}(L, \mathbb{C} \setminus G) \). The set \( M := \{ x \in \mathbb{C} \mid \text{dist}(x, L) \leq d/2 \} \) is compact, contained in \( G \) and \( L \subset M \). Define \( S := \{ z \in M \mid v(z) \geq \alpha \} \subset M \). We show that \( \partial L \subset S \). Indeed, if \( z \in \partial L \), there is a sequence \( (x_j)_j \subset G \) such that \( v(x_j) \geq \alpha \) for each \( j \in \mathbb{N} \) and \( x_j \to z \) as \( j \to \infty \). There is \( J \subset \mathbb{N} \) such that, for \( j \geq J \), \( \text{dist}(x_j, L) \leq |x_j - z| < d/2 \). If \( j \geq J \), then \( x_j \in M \) and \( v(x_j) \geq \alpha \), which means \( x_j \in S \). This implies \( z \in S \).

Now, if \( z \in K \subset L \) and \( f \in H(G) \), we can apply the maximum principle to get

\[
|f(z)| \leq \sup_{\zeta \in L} |f(\zeta)| \leq \sup_{\zeta \in \partial L} |f(\zeta)| \leq \sup_{\zeta \in S} |f(\zeta)|.
\]

This implies \( K \subset H^{co}(S) \). The conclusion now follows from Proposition 2.9, since we have shown that the assumption there is satisfied. \( \square \)

Corollary 2.11 Let \( v : G \to [0, \infty[ \) be a continuous weight on a domain \( G \subset \mathbb{C} \) such that \( G \setminus E_v \) is discrete, i.e., \( v \) has only isolated zeros. Then \( H_v^\infty(G) \) is a Banach space.

**Proof.** Denote by \( Z \) the discrete set of isolated zeros of \( v \) on \( G \). We show that the assumption of Proposition 2.10 is satisfied. Given a a compact set \( K \subset G \), the set \( K \subset Z \) is finite and the distance \( d := \text{dist}(K, Z \setminus K) \) is strictly positive, if \( Z \setminus K \neq \emptyset \). Otherwise, there will be a limit point of \( Z \) in \( K \). If \( Z \subset K \), set \( d = 1 \). Take a compact set \( L \) contained in \( G \), such that \( K \) is contained in the interior of \( L \), satisfying \( L \subset \{ z \in G \mid \text{dist}(z, K) \leq d/2 \} \). Then \( v(x) > 0 \) for each \( x \in \partial L \). Since \( v \) is continuous, \( \alpha := \min_{x \in \partial L} v(z) > 0 \), and we have \( K \subset L \) and \( \partial L \subset \{ z \in G \mid v(z) \geq \alpha \} \). The conclusion follows from Proposition 2.10. \( \square \)

Our next example is mentioned in [12, Example 3].

**Corollary 2.12** Let \( F \in H(G) \) be a non-zero holomorphic function on a domain \( G \subset \mathbb{C} \). If \( v(z) := |F(z)|, z \in G \), then \( H_v^\infty(G) \) is a Banach space.

**Proof.** This is a direct consequence of Corollary 2.11. \( \square \)

**Corollary 2.13** Let \( G = \mathbb{D} \) (resp. \( G = \mathbb{C} \)). Let \( v \) be a bounded radial weight on \( G \). The space \( H_v^\infty(G) \) is a Banach space if and only if \( E_v \) is not compact in \( G \), or equivalently if and only if there is an increasing sequence \( (r_k)_k \) in \( [0, 1] \) tending to 1 (resp. \( (r_k)_k \) in \( [0, \infty[ \) tending to \( \infty \)) such that \( v(r_k) > 0 \) for each \( k \in \mathbb{N} \).

In particular, if \( v(z) := |F(|z|)|, z \in G \), for a non-zero holomorphic function \( F \in H(G) \), then \( H_v^\infty(G) \) is a Banach space.
Proof. This is a direct consequence of Corollary 2.7 (2) and Proposition 2.10 considering compact sets of the form \( K = \{ z \in G \mid |z| \leq r \} \), \( \partial K = \{ z \in G \mid |z| = r \} \).

The conclusion for \( v(z) := |F(|z|)|, z \in G \), can be also deduced from Corollary 2.11.
\( \square \)

Let \( v \) be the weight on \( \mathbb{D} \) defined by \( v(z) := a_n > 0 \) if \( |z| = 1 - (1/n) \), and \( v(z) = 0 \) otherwise. Then \( H_v^\infty(G) \) is a Banach space by Corollary 2.13. Observe that the sequence \( (a_n) \) need not be bounded by Proposition 2.10. Similar examples can be obtained by replacing \( \mathbb{D} \) by \( \mathbb{C} \) and \( 1 - (1/n) \) by \( n, n \in \mathbb{N} \).

Proposition 2.14 Let \( F \in H(G) \) be a non-zero holomorphic function on a domain \( G \subset \mathbb{C} \). Define \( v(z) := 0 \) if \( F(z) = 0 \) and \( v(z) := 1/|F(z)| \) if \( F(z) \neq 0 \). Then \( H_v^\infty(G) \) is a Banach space that coincides with the set of all \( f \in H(G) \) such that there is \( C = C(f) > 0 \) with \(|f(z)| \leq C|F(z)|\) for each \( z \in G \).

Proof. The weight \( v \) is in general unbounded and not continuous, but it has isolated zeros. Then \( H_v^\infty(G) \) is a normed space by Proposition 2.1.

If \( f \in H(G) \) satisfies \(|f(z)| \leq C|F(z)|\) for each \( z \in G \), then \( f(z) = 0 \) whenever \( F(z) = 0 \). Moreover, \( v(z)|f(z)| \leq Cv(z)|F(z)| \) for each \( z \in G \), hence \( f \in H_v^\infty(G) \) and \(|f| \leq C \). On the other hand, if \( f \in H_v^\infty(G) \) satisfies \(|f| = D > 0 \), then \(|f(z)| \leq D|F(z)| \) if \( F(z) \neq 0 \). Since both \( f \) and \( F \) are continuous and the zeros of \( F \) are isolated, this inequality implies that \( f(z) = 0 \) whenever \( F(z) = 0 \). Therefore \( f \in H_v^\infty(G) \) if and only if there is \( C > 0 \) with \(|f(z)| \leq C|F(z)|\) for each \( z \in G \). Moreover, we have shown that the unit ball \( B_v^\infty \) of \( H_v^\infty(G) \) coincides with the set \( \{ f \in H(G) \mid |f(z)| \leq |F(z)| \text{ for all } z \in G \} \). This implies that \( B_v^\infty \) is bounded in \((H(G), \tau_{co})\), and \( H_v^\infty(G) \) is a Banach space by Proposition 2.3. \( \square \)

Remark 2.15 Observe that for the weight \( v \) on \( G \) considered in Proposition 2.14, we have \(|\delta_z||v| = 0 \) if \( F(z) = 0 \) (since each \( f \in H_v^\infty(G) \) vanishes on such \( z \in G \)), and \(|\delta_z||v| = |F(z)|\) if \( F(z) \neq 0 \). In fact, \( F \in H_v^\infty(G) \) and \(|F| = 1 \), thus \(|\delta_z||v| \geq |F(z)|\). Moreover, if \( f \in B_v^\infty \), then \(|f(z)| \leq |F(z)|\), which yields \(|\delta_z||v| \leq |F(z)|\). Therefore \(|\delta_z||v| = |F(z)|\) for each \( z \in G \). This implies that the weight \( \hat{v}(z) := 1/|\delta_z||v| \), associated with this particular weight \( v \) and constructed in the proof of Theorem 2.5 is not defined on the whole set \( G \) and is unbounded on the set on which it is defined. This shows that the assumption that the weight \( v \) is bounded is indeed necessary in Theorem 2.5.

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